MINIMIZATION OF THICKNESS OF NONUNIFORM ABSORBING LAYER WITH SPECIFIED MODULUS OF REFLECTION COEFFICIENT OF MONOCHROMATIC WAVE

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Minimization of the thickness of a nonuniform absorbing layer at specified modulus and phase of the reflected monochromatic wave is a matter of record [1]. Only the modulus of the reflection coefficient is generally prespecified in practice. The minimization problem applicable to that case is discussed in the present article.

1. Statement of Problem. Let a plane monochromatic wave impinge on a flat nonuniform absorbing layer whose index of refraction varies along the normal, in the same direction. Assuming, as in [1],

$$n^2 = 1 + (1 + i\eta)\, Q(\tau) \tag{1.1}$$

for the square of the relative refraction index, we will then have the following differential equations for the real and imaginary parts of the admittance $G = p + iq$:

$$\frac{dp}{d\tau} = \eta Q - 2pq, \qquad \frac{dq}{d\tau} = p^2 - q^2 - 1 - Q \tag{1.2}$$

Here G is the input admittance of the layer referred to the wave admittance $(1/\rho_0 c_0)$ of the medium from which the wave arrives; ρ_0, c_0 are the density of the medium and the speed of the longitudinal wave traversing the medium; $\tau = \omega c_0^{-1} x$ is the reduced thickness of the layer bounded by the planes $x = 0$ and $x = e$; ω is the angular frequency; x is the coordinate; η is specified positive constant; $Q(\tau)$ is a nonnegative function satisfying the definition

$$0 \leqslant Q(\tau) \leqslant N \tag{1.3}$$

The boundary conditions imposed on values of the phase coordinates p and q are

$$p(0) = p^\circ, \qquad q(0) = q^\circ \qquad (\tau = 0) \tag{1.4}$$

$$\Phi_e \equiv \frac{(1 - p_e)^2 + q_e^2}{(1 + p_e)^2 + q_e^2} - |\beta_e|^2 = 0 \qquad (\tau = \tau_e) \tag{1.5}$$

where $G(0) = p^\circ + iq^\circ$; $G(\tau_e) = G_e = p_e + iq_e$ are the respective input admittances of the load and of the nonuniform layer, τ_e is the reduced thickness of the layer bounded by the planes $x = 0$ and $x = e$, and $|\beta_e|$ is the assigned value of the modulus of the reflection coefficient.

The optimization problem is formulated as follows: to determine the phase coordinates $p(\tau)$ and $q(\tau)$, satisfying equations (1.2) and initial conditions (1.4), and also the control function $Q(\tau)$ obeying inequality (1.3), such that the reduced thickness τ_e of the layer will be minimized when constraint (1.5) is satisfied.

2. Extremal Partial Arcs. The optimum problem formulated [(1.2)-(1.5)] differs from the one considered in [1] solely in the form of the boundary condition (1.5). Hence the principal inferences drawn in [1] remain valid here. The optimum control can assume only the boundary values: $Q = 0$, or else $Q = N$.

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Corresponding to those values, the optimum admittance hodograph is found from solutions of the differential equations (1.2)

$$p + iq = \frac{p_0 + iq_0 \mp i\,(1 + (1 + i\eta)\,Q)^{1/2}\,\mathrm{tg}\,[\tau\,(1 + (1 + i\eta)\,Q)^{1/2}]}{1 \mp i(p_0 + iq_0)\,(1 + (1 + i\eta)Q)^{-1/2}\,\mathrm{tg}\,[\tau\,(1 + (1 + i\eta)\,Q)^{1/2}]} \tag{2.1}$$

where (p_0, q_0) is a point of the admittance plane corresponding to the onset of the change in τ on any subinterval within which $Q(\tau)$ retains its value unaffected. The minus sign in front of i in Eq. (2.1) corresponds to direct integration, the plus sign to inverse integration. The differential equations for the auxiliary phase coordinates (Lagrangian multipliers) are

$$\frac{d\lambda_p}{d\tau} = 2q\lambda_p - 2p\lambda_q, \qquad \frac{d\lambda_q}{d\tau} = 2p\lambda_p + 2q\lambda_q \tag{2.2}$$

The switching function, whose zeros determine the "moments" of the stepwise variation in control $Q(\tau)$ from 0 to N (and vice versa), is expressed in terms of the Lagrangian multipliers

$$K = \eta\lambda_p - \lambda_q \tag{2.3}$$

The control value is maximized on intervals of the optimum admittance hodographs such that $K > 0$, and minimized on those intervals where $K < 0$. The first integral of Eqs. (1.2) and (2.2) is

$$H_\lambda = KQ - 2pq\lambda_p - (1 + q^2 - p^2)\,\lambda_q = 1 \tag{2.4}$$

In the region $\Phi = 2pq + \eta\ (1 + q^2 - p^2) > 0$, switching (i.e., a stepwise change in the control response) can occur only from minimum to maximum, and the reverse holds for the $\Phi < 0$ region.

Equation (1.5) leads, in accordance with [1] [formula (2.4)] to the following boundary conditions for the Lagrangian multipliers:

$$\lambda_p{}^e = \lambda_p(\tau_e) = \chi_e \frac{4\varphi_e}{[(1 + p_e)^2 + q_e{}^2]^2}$$
$$\lambda_q{}^e = \lambda_q(\tau_e) = -\chi_e \frac{8p_e q_e}{[(1 + p_e)^2 + q_e{}^2]^2} \tag{2.5}$$

where χ_e is an unknown constant, $\varphi_e = 1 + q_e{}^2 - p_e{}^2$. The existence of a relationship between values of the phase coordinates p_e, q_e, and the auxiliary functions $\lambda_p{}^e$, $\lambda_q{}^e$ at $\tau = \tau_e$, and the absence of any such relationship at $\tau = 0$, allow us to infer that (simultaneous) integration of Eqs. (1.2) and (2.2) in the inverse direction should not be attempted. As a result of inverse integration of equations (1.2), we have the expression (2.1) with the plus sign in front of the imaginary unit. Integration of equations (2.2) can be performed with the aid of the auxiliary complex function $\lambda = \lambda_q + i\lambda_p$, for which the two equations (2.2) are written in the form

$$d\lambda/d\tau = -2iG\lambda \tag{2.6}$$

The solution of Eq. (2.6) will be

$$\lambda = \exp\Big(-2\int_0^\tau q\,d\tau\Big)\Big[\lambda_q{}^\circ \cos 2\int_0^\tau p\,d\tau - \lambda_p{}^\circ \sin 2\int_0^\tau p\,d\tau + i\Big(\lambda_p{}^\circ \cos 2\int_0^\tau p\,d\tau + \lambda_q{}^\circ \sin 2\int_0^\tau p\,d\tau\Big)\Big] \tag{2.7}$$

or, by using the expression for $G = p + iq$ [Eq. (2.1)],

$$\lambda = \frac{\lambda_q{}^\circ + i\lambda_p{}^\circ}{1 + \mathrm{tg}^2\,[\tau\,(1 + (1 + i\eta)\,Q)^{1/2}]}\left(1 + i\,\frac{(p_0 + iq_0)\,\mathrm{tg}\,[\tau\,(1 + (1 + i\eta)\,Q)^{1/2}]}{(1 + (1 + i\eta)\,Q)^{1/2}}\right)^2 \tag{2.8}$$

Here $(\lambda_p{}^\circ, \lambda_q{}^\circ)$ is a point on the auxiliary plane $\lambda_p\lambda_q$ corresponding to the onset of variation in τ on any subinterval within which $Q(\tau)$ retains its value unaltered.

3. Switching Curves. In order to solve the optimum problem formulated above, we have to construct a family of switching curves on the phase plane pq, i.e., we have to find the geometrical locus of points determined using Eq. (2.1), at those values of τ_j (switching moments) at which the switching function (2.3) vanishes. The initial data for constructing the family of curves will be the finite values of the phase coordinates (p_e, q_e) lying on a circle of specified modulus of the reflection coefficient (Fig. 1). The qualitative features of the situation of interest are such. By utilizing Eqs. (2.3) and (2.7) we find, for the switching function

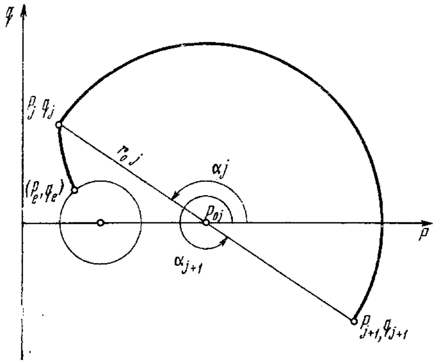

Fig. 1

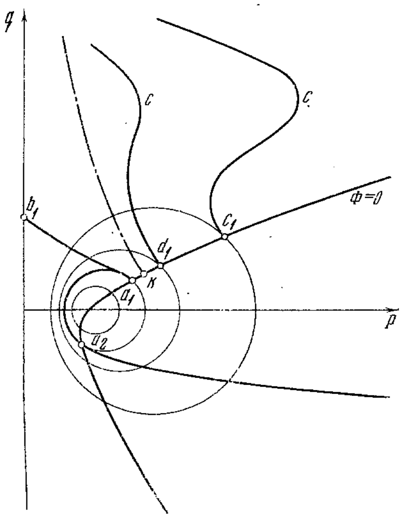

Fig. 2

$$K = \exp\left(-2\int_0^{\tau} q\,d\tau\right)\left[K_0 \cos 2\int_0^{\tau} p\,d\tau + (\lambda_p{}^\circ + \eta\lambda_q{}^\circ)\sin 2\int_0^{\tau} p\,d\tau\right] \qquad (3.1)$$

The values of the switching moments are determined from the equations

$$\operatorname{tg} 2\int_0^{\tau} p\,d\tau = -\frac{K_j^\circ}{(\lambda_p{}^\circ)_j + \eta(\lambda_q{}^\circ)_j} \quad (j = 1, \ldots, n) \qquad (3.2)$$

Since the variable K_j° vanishes (by definition) for the second and subsequent switching curves, for these we have

$$\int_0^{\tau_j} p\,d\tau = \frac{\pi}{2} \quad (j = 2, \ldots, n) \qquad (3.3)$$

For those intervals of the optimum admittance hodographs such that Q = 0, with the aid of the new variable α (see Fig. 1), we have

$$p = p_j^\circ + r_j^\circ \cos\alpha, \qquad q = r_j^\circ \sin\alpha \qquad (3.4)$$
$$p_j^\circ = \frac{p_j^2 + q_j^2 + 1}{2p_j}, \qquad r_j^\circ = [(p_j^\circ)^2 - 1]^{1/2}$$

and so

$$\alpha_{j+1} - \alpha_j = \pi \qquad (3.5)$$

Equations (3.4) and (3.5) show that these intervals of the admittance hodographs are half-circles with radii r_j° and centers lying on the p axis. In the case of intervals of the admittance hodographs such that Q = N (and $\eta \neq 0$), equations (3.2) are solved numerically, or approximately. When interested in τ_1 values, we can state for the first switching curve

$$\operatorname{tg} 2\int_0^{\tau_1} p\,d\tau = \frac{\Phi_e}{2\eta p_e q_e - \varphi_e} \qquad (3.6)$$

For points of a circle of specified modulus of reflection coefficient close to the point a_1 (Figs. 2 and 3), the right-hand member of Eq. (3.6) will be infinitesimally small, so that

$$\int_0^{\tau_1} p\,d\tau \approx \frac{1}{2}\,\frac{\Phi_e}{2\eta p_e q_e - \varphi_e} \qquad (3.7)$$

The τ_1 values will also be small in that case; the switching curve begins at point a_1. As G_e becomes further removed from point a_1, the τ_1 values will grow. Point a_2 (Fig. 2) will be the extreme point on the specified circle, since the extremal admittance hodographs emerging from the $\Phi < 0$ region will not be optimal (the minimum thickness for these loci will be greater by an amount corresponding to that portion of the hodograph lying within the circle of the specified modulus of the reflection coefficient).

The position of the "final" point on the switching curve is not quite so simple to establish. Here we set up a family of clockwise-unwinding spiraling admittance hodographs (winding from the focus G_∞ = $[1 + (1 + i\eta)N]^{1/2}$) "threading" that portion of the specified circle between points a_1 and a_2. Some of the hodographs (of those emerging from the portion of the circle adjacent to point a_1) will pass through the point at which $dq/dp = \infty$ and will deflect in the direction of increasing phase coordinate p. The others (emerging from the portion of the circle adjacent to the point a_2) will deflect to the opposite side and then intersect the p = 0 axis and continue into the negative half-plane p < 0.

Since processes associated with propagation of waves through absorbing media are described by phase coordinates on the positive half-plane, the p = 0 axis can be considered the boundary of that region of admittance values. Consequently, it is precisely on that axis that the final point of the switching curve will

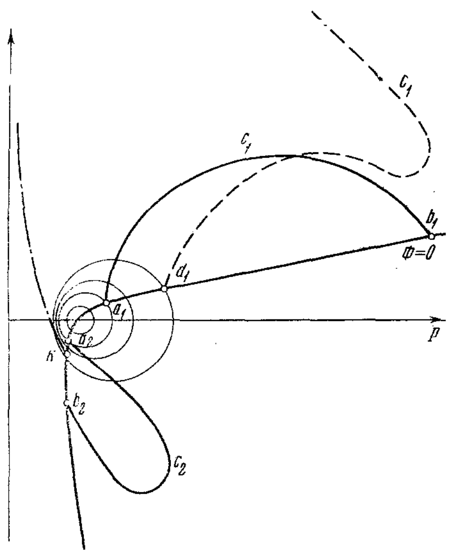

Fig. 3

lie, if that point belongs to any of the hodographs deflecting in the direction of the negative half-plane $p < 0$ and intersecting that axis.

If this switching point lying on the $p = 0$ axis is found in the inverse integration of equations (1.2) proceeding from some point (p_e, q_e), we shall find, by continuing the integration process along the axis, that the switching function will be identically zero everywhere on that axis, all the way out to the infinitely remote point on the axis. This is inferred from Eq. (3.1) when the substitution $p = K_0 = 0$ is made and is a consequence of the homogeneity of the differential equations for Lagrangian multipliers.

This is quite understandable once we bear in mind that the nonvanishing real part of the admittance of an arbitrary medium, with optimum matching of purely imaginary loads to the medium, will correspond to switching points lying solely on the $p = 0$ axis. But if not a single one of the hodographs emerging from points on the circle of specified modulus of the reflection coefficient intersects the $p = 0$ axis, then the final point on the switching curve (corresponding to point b_1) will then lie on the hyperbola $\Phi = 0$. (The final point cannot lie in the $\Phi < 0$ region, since the $Q = 0$ value corresponds to the optimum hodograph in that region.) This pattern will hold in the case of low η values and reasonably low N values. As η and N increase, a family of optimum admittance loci intersecting the $p = 0$ axis will show up. When the η and N values are fixed, the question of whether a circle of the specified modulus of the reflection coefficient corresponds to some family of optimum hodographs intersecting the $p = 0$ axis will be easy to solve. All that will be required is to plot the "limiting" locus (hodograph) intersecting the $p = 0$ axis at infinitely large q values in accordance with the formula*

$$p_e + iq_e = i\,(1 + (1 + i\eta)\,N)^{1/2}\,\mathrm{ctg}\,[\tau\,(1 + (1 + i\eta)\,N)^{1/2}] \tag{3.8}$$

If the limiting locus proceeds to the left of and below point a_2, there will be no optimum admittance hodographs intersecting the $p = 0$ axis, so that the first switching curve will be depicted by the curve $a_1c_1b_1$ (Fig. 3). We can infer from Eqs. (2.8) and (3.1) that the switching point for the limiting hodograph (independently of the point G_e) will be the infinitely remote point (a singularity for the function λ). This last point signifies that if the limiting hodograph passes through the point a_2 or above it and to the right of it, the final point on the switching curve will be found on the $p = 0$ axis. In that case the switching curve will be depicted by the curves a_1b_1, d_1c, and c_1c (Fig. 2).

In order to construct the second switching curve, we have to draw an arc of the semicircle centered at $p_1°$, in accordance with Eqs. (3.4) and (3.5), from each point of the first switching curve, the radius being $r_1°$. The locus of the termini of these arcs will then yield the second switching curve. The initial point of the first switching curve (a_1) will be mapped into point (a_2) corresponding now to the origin of the second switching curve. But if the final point on the second switching curve lies on the $p = 0$ axis, the corresponding final point on the second switching curve will be found at the infinitely remote point for which, however, the value of the imaginary coordinate will be equal to the value, with sign reversed, of the imaginary coordinate of the final point on the first switching curve.

In that case, as is clear from Fig. 2, the maximum number of uniform layers must not exceed three in the case of optimum matching of load to an arbitrary complex admittance (on the level of the specified modulus of the reflection coefficient). Now let the final point on the first switching curve lie on the hyperbola $\Phi = 0$ (point b_1), i.e., let the limiting admittance hodograph proceed below and to the left of point a_2. Then point b_1 will be reflected into point b_2, the final point for the second switching curve (curve $a_2c_2b_2$). But if the limiting hodograph intersects the hyperbola $\Phi = 0$ on the interval a_2b_2 in that case, then we shall find, when the second switching curve is mapped, that the final point on the third switching curve will lie on the $p = 0$ axis.

The corresponding final point on the fourth switching curve will then proceed out to the infinitely remote point. In that case the maximum number of uniform layers will increase to five in the case of

*This locus will intersect the hyperbola $\Phi = 0$ at point K in Figs. 2 and 3.

optimum matching of an arbitrary complex load. And if the limiting admittance hodograph proceeds below and to the left of point b_2, we can, in similar fashion, establish the maximum number of uniform layers required for optimum matching. By constructing the complete pattern of switching curves, we can use Eq. (2.1) to compute the minimum possible thickness and also other parameters of the optimum nonuniform layer.

LITERATURE CITED

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